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Reaching Conditions for Variable Structure Control with Output Feedback

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A design technique for satisfying the reaching condition when applying variable structure control to a linear system with output feedback is described. The control consists of a linear feedback term and a discontinuous feedback term. By combining the linear term with the discontinuous term, very good disturbance rejection and robustness can be achieved with a lower control effort than would be required from either control when used independently. The linear gain and the discontinuous gain are designed so that the sliding surface exists and is globally reachable. Furthermore, the time spent in the reaching phase can be reduced by proper selection of a scalar parameter that is imbedded in the linear-gain matrix. A specific algorithm is given along with a numerical example.

I. Introduction

VARIABLE structure control (VSC) is a robust nonlinear control strategy employing feedback of a discontinuous signal. The discontinuous feedback forces the trajectory to remain on a sliding surface, during which time the system exhibits very good disturbance rejection and robustness. Most of the previous work in VSC employed either full-state or estimated-state feedback, which may be impractical or overly complicated to implement. Two alternatives that are easier to implement are static output feedback and dynamic output feedback. Static output feedback was first introduced by Heck and Ferri,[1] and design methods were considered later.[2–10] Dynamic feedback[11,12] is harder to design but generally exhibits better performance than static feedback.

This paper addresses the design of a control for the reaching phase, using static output feedback. The control consists of a linear feedback term plus a discontinuous term, which guarantee that the sliding mode exists and is globally reachable under a very mild restriction. This paper expands on the design method first developed by Yallapragada et al.[2] Other papers[1,3–10] have also considered the reaching phase design. Several of the papers[1,3–6] show that the sliding mode exists but do not show it to be globally reachable. Teixeira[7] shows that the system is globally asymptotically stable under certain conditions but does not give a constructive method of obtaining the control. Zak and Hui[8] require an assumption on the existence of a particular matrix, but no mention is made on how to find the matrix; also, the assumption appears to be overly restrictive. Two recent papers[9,10] also give constructive means of finding a globally stabilizing control; however, these controls are designed using methods other than those given here and will yield larger control efforts for some systems.

By combining a linear feedback with a discontinuous feedback, it is shown below that very good robustness and disturbance rejection can be achieved without the high control effort that would be required from either control when used independently. The method developed in this paper contains four scalar design parameters that can be chosen to achieve good disturbance rejection and robustness without requiring unduly large values of the control signal. For design flexibility, the analysis is developed for two forms of the discontinuous control: $u_d = s/\|s\|$, which is fairly standard, and $u_d = s/\|s\|^2$, which is not standard. The properties of the nonstandard form are explored and its advantages over the use of the more common discontinuous law are shown. These advantages may extend to the more general design of variable structure controllers. Note that the developed design method is also applicable when dynamic feedback is used by first augmenting the compensator dynamics to the plant, resulting in a static output feedback formulation.

The problem formulation is given below and the design methods are developed in Sec. II. Some background information on positive real systems, which are utilized in this paper, also is given in Sec. II, along with a discussion of the robustness properties of the control. Section III contains a specific algorithm followed by a numerical example and a discussion of practical design considerations.

Consider a linear time-invariant system described by

$$\dot{x} = Ax + Bu \qquad y = Cx \tag{1}$$

where $x \in R^n$, $y \in R^p$, $u \in R^m$, and $p > m$. It is assumed that the system is observable and controllable, that CB is full rank, and that the open-loop system is minimum phase. Note that many papers on robust output feedback require that CB be full rank.[1–12,15,16,18] A linear switching surface is defined by $s = Gy$, where $G \in R^{m\times p}$. The control is chosen to be

$$u = Ny - \alpha(GCB)^{-1}u_d \tag{2}$$

where $N \in R^{m\times p}$ is a constant gain matrix, α is a scalar, and two forms of u_d are considered in this paper: $u_d = s/\|s\|$, which is a common form of the discontinuous law, and $u_d = s/\|s\|^2$, which is not commonly used but allows more flexibility in the design to reduce the overall control effort. The sliding mode is found from the equivalent control method to be given by

$$\dot{x} = [A - B(GCB)^{-1}GCA]x \tag{3}$$

This paper assumes that G has already been chosen such that GCB is invertible and Eq. (3) has desired characteristics; see Refs. 8–10 and 12 for specific design methods. It is also assumed that the system defined by the triple (A, B, GC) is observable and controllable. (Controllability is guaranteed because (A, B, C) must be controllable.) This is a very mild restriction.

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For simplicity in the following development, note that a nonsingular transformation of switching surface does not change the sliding-mode dynamics.[16] If a particular switching surface $s_1 = G_1 y$ is chosen, it can be transformed to $s = Gy$, where $G = (G_1 CB)^{-1} G_1$. Therefore, without loss of generality, we can assume that $GCB = I$.

The reaching condition is defined by $s^T \dot{s} < 0$. For this system, this condition reduces to

$$s^T \dot{s} = s^T GC(A + BNC)x - \alpha \|s\| < 0 \tag{4}$$

if $u_d = s/\|s\|$. The last term in Eq. (4) is replaced by $-\alpha$ if $u_d = s/\|s\|^2$.

Note that to reduce chattering in an actual implementation, the discontinuous controls often are replaced by a smoothed control of the form $s/(\|s\| + \delta)$ or $s/(\|s\|^2 + \delta)$ where $\delta > 0$ is small. This creates a small boundary layer about the switching surface in which the system trajectory will remain, as opposed to ideal sliding where the trajectory remains on the switching surface. Moreover, the control $s/\|s\|^2$ should not be implemented without the smoothing because it is unbounded as $s \to 0$.

II. Design Methods

This section shows how to choose N and α in Eq. (2) to satisfy the reaching condition in a region defined by $\{x: \|x_k\| \leq \Omega\}$, where x_k is the component of x in the null space of GC. The developed method makes the best use of the measurement y to increase the size of the region in which the reaching condition $s^T \dot{s} < 0$ is satisfied. The measurement vector was used similarly in Refs. 3 and 9 to develop different control laws. It is also shown here how the reaching time can be reduced by proper selection of a scalar parameter imbedded in N. In addition, it is shown that the closed-loop system is globally asymptotically stable; therefore, the system trajectory will always enter the region in which the reaching conditions hold. Hence, the sliding surface is globally reachable. Finally, the robustness properties of the control are explored.

A. Reaching Condition

Define the singular value decomposition of GC as $GC = U\Sigma V_1^T$, where $U \in R^{m\times m}$, $\Sigma \in R^{m\times m}$, and $V_1 \in R^{n\times m}$.

Theorem 1. For the control law in Eq. (2) with $u_d = s/\|s\|$, choose

$$N = -\left(\gamma I + GCAV_1\Sigma^{-1}U^T\right)G \tag{5}$$

where $\gamma > 0$ is a scalar. If α is chosen to satisfy $\alpha > \|GCA\|\Omega$, where $\Omega > 0$, then the reaching condition $s^T \dot{s} < 0$ is satisfied for $\{x: \|x_k\| \leq \Omega\}$.

Proof. Substitution of N into the reaching condition in Eq. (4) yields

$$s^T \dot{s} = s^T GC\left(A - BGCAV_1\Sigma^{-1}U^T GC\right)x - \gamma s^T s - \alpha\|s\| \tag{6}$$

If x is decomposed as $x = x_k + x_p$, where $x_k \in \mathcal{N}(GC)$, $x_p \in \mathcal{N}^{\perp}(GC)$, then $GCx = GCx_p$. Simplify the first term on the right-hand side using $GCB = I$, and then decompose it as follows:

$$s^T GCA\left(I - V_1\Sigma^{-1}U^T GC\right)x = s^T GCAx_k$$
$$+ s^T GCA\left(I - V_1\Sigma^{-1}U^T GC\right)x_p \tag{7}$$

If $GC = U\Sigma V_1^T$, then $x_p = V_1\beta$ for some vector β. Using $V_1^T V_1 = I$ and $U^T U = I$, the second term on the right-hand side of Eq. (7) is zero, and so the expression in Eq. (6) reduces to

$$s^T \dot{s} = s^T GCAx_k - \gamma s^T s - \alpha\|s\| \tag{8}$$

If $\Omega \geq \|x_k\|$, an upper bound for the expression in Eq. (8) is given by

$$s^T \dot{s} \leq \|s\|\|GCA\|\Omega - \gamma s^T s - \alpha\|s\| \tag{9}$$

If $\alpha \geq \|GCA\|\Omega$, the reaching condition is satisfied. □

Corollary 1. For the control law in Eq. (2), if $u_d = s/\|s\|^2$, N is chosen as in Eq. (5), where $\gamma > 0$, and if α is chosen to satisfy $\alpha > \|GCA\|^2\Omega^2/4\gamma$, then the reaching condition $s^T \dot{s} < 0$ is satisfied for $\{x: \|x_k\| \leq \Omega\}$.

Proof. For this value of u_d, Eq. (9) is replaced by

$$s^T \dot{s} \leq \|s\|\|GCA\|\Omega - \gamma\|s\|^2 - \alpha \tag{10}$$

Let the right-hand side of Eq. (10) be defined as

$$z \triangleq -\gamma\|s\|^2 + \|GCA\|\Omega\|s\| - \alpha \tag{11}$$

Then Eq. (11) represents a parabola that is concave down with vertex at

$$(\|s\|, z) = \left[\frac{\|GCA\|\Omega}{2\gamma}, \frac{\|GCA\|^2\Omega^2}{4\gamma} - \alpha\right] \tag{12}$$

To ensure that $z < 0$, choose $\alpha > \|GCA\|^2\Omega^2/4\gamma$. □

For the control law $u_d = s/\|s\|$, a larger choice of Ω necessitates a larger choice of the discontinuous control gain α. This is not the case for $u_d = s/\|s\|^2$, because of the γ term in Eq. (12). The larger γ, the smaller α needs to be. Hence, there is a tradeoff between the size of the linear control gain N and the discontinuous gain α. An alternative expression for α that might yield a smaller value is given in Corollary 2.

Corollary 2. Let $L = C^T G^T(GCA + NC)$. If $u_d = s/\|s\|^2$ in Eq. (2) and N is chosen as in Eq. (5), where $\gamma > 0$, and if α is chosen such that $\alpha > \lambda_{\max}(L + L^T)\Omega^2/2$, then the reaching condition $s^T \dot{s} < 0$ is satisfied for $\{x: \|x\| \leq \Omega\}$.

Proof. The reaching condition for this control is given by

$$s^T \dot{s} = s^T GC(A + BNC)x - \alpha < 0 \tag{13}$$

Letting $s = GCx$, and noting that $GCB = I$, yields

$$\begin{aligned} s^T \dot{s} &= x^T C^T G^T(GCA + NC)x - \alpha \\ &= \tfrac{1}{2}x^T(L + L^T)x - \alpha \\ &\leq \tfrac{1}{2}\lambda_{\max}(L + L^T)\|x\|^2 - \alpha \end{aligned} \tag{14}$$

It follows that $s^T \dot{s} < 0$ in the region $\{x: \|x\| \leq \Omega\}$ if α is chosen such that $\alpha > \lambda_{\max}(L + L^T)\Omega^2/2$. □

With a proper choice of γ, the reaching time (i.e., the time spent in the reaching phase) can be reduced. This is important because the robustness properties afforded by the control law only hold during the sliding phase.

Corollary 3. The trajectory is exponentially stable to the switching surface, i.e., $s(t)$ decays as

$$\|s(t)\| \leq e^{-(\gamma t/2)}\|s(0)\| \tag{15}$$

Proof. Suppose V is a Lyapunov function for a system. If $\dot{V} \leq -\gamma V$, then the system is quadratically stable to the switching surface and $V(t) \leq e^{-\gamma t}V(0)$ (Ref. 10). Let $V = s^T s$ and $\dot{V} = s^T \dot{s}$. From Eq. (9) or (10), it is seen that if α is chosen as suggested in Theorem 1 or Corollary 1, then $s^T \dot{s} \leq -\gamma s^T s$. The result in Eq. (15) follows directly. Note that the exponential decay bound is conservative because $\dot{V}$ is even more negative than $-\gamma V$. □

As mentioned in the Introduction, the discontinuous controls often are approximated by continuous function for implementation. Replacing $s/\|s\|$ by $s/(\|s\| + \delta)$ and $s/(\|s\|^2 + \delta)$ still results in satisfying $s^T \dot{s} < 0$ for a large enough $\|s\|$; consequently, the sliding behavior exists in a region about the switching surface rather than being restricted to the switching surface.

B. Global Asymptotic Stability

In the previous theorem and corollaries, the scalar parameter γ in the gain N defined in Eq. (5) simply needed to be positive. Next, it is shown that $\gamma > 0$ can always be chosen to make the origin globally asymptotically stable. As a result, the system trajectory will always enter the region $\{x: \|x_k\| \leq \Omega\}$. Hence, the sliding surface is globally reachable. The proof is based on the concepts of strictly positive real (SPR) systems. The design of VSC in SPR systems was addressed briefly in the paper by Teixeira,[7] where the emphasis was on developing necessary and sufficient conditions for turning an output feedback system into an SPR system (a specific control was not derived). Yallapragada et al.[2] used SPR concepts to design a variable-structure output feedback control that guarantees global asymptotic stability; this section extends and expands the method. Edwards and Spurgeon[10] independently developed a similar method to guarantee global asymptotic stability. Their method, which first transforms the system into a canonical form, may result in larger linear control gains than achievable with the method shown here.

The stability of system (1) can be examined by determining the stability of the squared-down system, which has a new output vector:

$$\dot{x} = Ax + Bu \qquad \tilde{y} = GCx \tag{16}$$

Note that the number of outputs m equals the number of inputs. It is assumed that this system is observable and controllable. This system is SPR if it satisfies either of the following conditions in Lemma 1.

Lemma 1.[13] Given an asymptotically stable minimal realization defined by the triple (A, B, GC) with transfer function matrix $H(s) = GC(sI - A)^{-1}B$, then conditions 1 and 2 are equivalent:

1a) $H(j\omega) + H^T(-j\omega) > 0 \ \ \forall \omega \in R$

and

1b) $\lim_{\omega\to\infty} \omega^2[H(j\omega) + H^T(-j\omega)] > 0 \quad \forall \omega \in R$

2) There exist a symmetric $P > 0$, $Q \in R^{m\times n}$, and $\varepsilon > 0$ such that

$$A^TP + PA = -Q^TQ - 2\varepsilon P \qquad PB = C^TG^T \tag{17}$$

It will be assumed in the following theorem that system (16) is SPR. Note that a necessary condition for SPR is that the matrix GCB must be symmetric and positive-definite.[14] With the transformation on the switching surface given in Sec. I.A, this condition is satisfied because $GCB = I$. The case where system (16) is not SPR is handled afterward with a minor modification.

Theorem 2. System (16) is globally asymptotically stable to the origin for the control law defined in Eq. (2), where N is defined as in Eq. (5), $\alpha > 0$, and $u_d = s/\|s\|$, if the system is SPR and if γ is chosen such that $2\gamma \ge -\lambda_{\min}(GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^TA^TC^TG^T)$.

Proof. Let $V(x) = x^TPx$ be a Lyapunov function candidate for the closed-loop system (16). Then,

$$\dot{V} = (Ax + Bu)^TPx + x^TP(Ax + Bu) \tag{18}$$

Substitute the control defined in Eq. (2) into Eq. (18) to yield

$$\dot{V} = x^T[PA + PBNC + (PA + PBNC)^T]x - 2\alpha x^TPBu_d \tag{19}$$

where N is defined in Eq. (5). Substitute $PA + A^TP = -Q^TQ - 2\varepsilon P$ and $u_d = s/\|s\|$ in Eq. (19) to yield

$$\dot{V} = -x^T(Q^TQ + 2\varepsilon P)x + x^T[PBNC + (PBNC)^T]x - (2\alpha/\|s\|)x^TPBs \tag{20}$$

Using $PB = C^TG^T$ from Eq. (17), the last term simplifies to $-2\alpha\|s\|$. The second term can be simplified using Eq. (6) and $PB = C^TG^T$:

$$x^TPBNCx = -x^T\big(\gamma PBGC + PBGCAV_1\Sigma^{-1}U^TGC\big)x = -s^T\big(\gamma I + GCAV_1\Sigma^{-1}U^T\big)s \tag{21}$$

Using Eq. (21), the expression in Eq. (20) then can be written as

$$\dot{V} = -x^T(Q^TQ + 2\varepsilon P)x - s^T\big(2\gamma I + GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^TA^TC^TG^T\big)s - 2\alpha\|s\| \tag{22}$$

The first and last terms are negative, and the second term is negative if γ is chosen such that

$$\lambda_{\min}\big(2\gamma I + GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^TA^TC^TG^T\big) = 2\gamma + \lambda_{\min}\big(GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^TA^TC^TG^T\big) \ge 0 \quad \square$$

Stability also can be shown for other forms of the control u_d, including $u_d = s/\|s\|^2$, $s/(\|s\|+\delta)$, and $s/(\|s\|^2+\delta)$, where $\delta > 0$.

Corollary 4. If $u_d = s/\|s\|^2$ or $u_d = s/(\|s\|^2 + \delta)$ or $u_d = s/(\|s\|+\delta)$, and the rest of the conditions in Theorem 2 are satisfied, then the closed-loop system in Eq. (16) is globally asymptotically stable to the origin.

Proof. Using $V = x^TPx$ results in an expression for dV/dt as given in Eq. (22), where the last term, $-2\alpha\|s\|$, is replaced by -2α or $-2\alpha\|s\|^2/(\|s\|^2 + \delta)$ or $-2\alpha\|s\|^2/(\|s\| + \delta)$ for $u_d = s/\|s\|^2$ or $u_d = s/(\|s\|^2 + \delta)$ or $u_d = s/(\|s\| + \delta)$, respectively. Because each of these terms is negative, the stability result remains intact. $\square$

In the above theorem and corollaries, it was assumed that the system defined in Eq. (16) is SPR. If it is not SPR, then the system can be reformulated into a system that is SPR with a large enough choice of the scalar γ.

Theorem 3. If the sliding-mode dynamics are stable and if $GCB = I$, then there exists a scalar $\gamma_1^* > 0$ such that applying the linear feedback $u = -\gamma_1\tilde{y}$ renders the closed-loop system defined by the triple $(A - \gamma_1 BGC, B, GC)$ SPR for all $\gamma_1 \ge \gamma_1^*$.

Proof. First, note that (A, B, GC) is minimum phase because the sliding-mode dynamics are stable.[17] Because (A, B, GC) is minimum phase and $GCB = I$, a procedure similar to that used in Ref. 15 can be used to prove Condition 1a of Lemma 1. For a system given by (A, B, C), Condition 1b can be shown to be equivalent to requiring that $[CAB + (CAB)^T] < 0$ (Ref. 14). For the triple $(A - \gamma BGC, B, GC)$, this condition becomes

$$GC(A - \gamma BGC)B + [GC(A - \gamma BGC)B]^T = GCAB + (GCAB)^T - 2\gamma I < 0$$

which is satisfied for sufficiently large γ. $\square$

Thus, if the system in Eq. (16) is not SPR, then the linear feedback term in Eq. (2) can be decomposed into a part that ensures SPR and a remaining part. Let $\gamma = \gamma_1 + \gamma_2$ in the expression for N given in Eq. (5), where γ_1 is chosen as required in Theorem 3. The required feedback to ensure that the system is SPR is $-\gamma_1\tilde{y} = -\gamma_1 Gy$. With this feedback, the triple (A, B, GC) becomes $(A - \gamma_1 BGC, B, GC)$, which is SPR. Hence, the following modification to Theorem 2 can be made for the case of a system (16) that is not initially SPR.

Theorem 4. The system given in Eq. (16) is globally asymptotically stable to the origin for the control given in Eq. (2), where u_d is chosen to be $s/\|s\|$, or $s/\|s\|^2$, or $s/(\|s\| + \delta)$ or $s/(\|s\|^2 + \delta)$; N is given by Eq. (5); and $\gamma = \gamma_1 + \gamma_2$ is chosen such that $\gamma > 0$ and $(A - \gamma_1 BGC, B, GC)$ is SPR and $\gamma_2 \ge -0.5\lambda_{\min}(GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^TA^TC^TG^T)$.

The proof follows that of Theorem 2.

Note that the form for N in Eq. (5) was chosen so that the linear feedback makes the best use of the part of the state x that is not in the null space of GC; hence it helps to stabilize the system to the switching surface. The corresponding reaching condition $s^T\dot{s} < 0$ is satisfied in the region $R_1 = \{x: \|x_k\| \le \Omega\}$, where x_k is the component of x in $\mathcal{N}(GC)$. However, the use of Theorems 2 and 3 and Corollaries 4 and 5 may result in large gains for N. Modifying the control law may yield a reduction in the gains at the expense of decreasing the size of the region in which the reaching condition holds. In particular, suppose the gain in Eq. (2) is chosen so that $N = -\gamma G$, where $\gamma > 0$. Then it can be shown that the results of Theorem 1 and Corollaries 1–3 hold for the region $R_2 = \{x: \|x\| \le \Omega\}$, where $R_2 \subset R_1$. The results of Theorem 2 and Corollary 4 hold if γ is chosen such that $\gamma \ge 0$. The results of Theorem 4 hold if γ_2 is chosen such that $\gamma_2 \ge 0$.

C. Robustness Properties

The results in the above theorems can be extended for application when the actual system has a disturbance, uncertainty, or nonlinearity $h(x)$:

$$\dot{x} = Ax + Bu + h \qquad y = Cx$$

Consider the case where h is bounded and matched, that is, $h = Be$, where $\|e\| \le \kappa(y, t)$. Then, the control $u = Ny - \alpha s/\|s\|$ satisfies the reaching condition where N is chosen as in Eq. (5) and α is chosen such that $\alpha > (\|GCA\|\Omega + \kappa)$. This same control also can be shown to be globally asymptotically stable. This control with a smoothing term, $u_d = s/(\|s\| + \delta)$, creates a boundary layer about the switching surface. The corresponding response is ultimately bounded, that is, $\|x\| \leq \theta < \infty$ for large enough t, with a bound θ that is made smaller by decreasing δ.

The control $u = Ny - \alpha s/\|s\|^2$ satisfies the reaching condition if N is chosen as in Eq. (5) and α is chosen such that

$$\alpha > \min\left\{(\kappa + \|GCA\|\Omega)^2/4\gamma,\ \lambda_{\max}(L + L^T)\Omega^2/2 + \kappa\|GC\|\Omega\right\}$$

where $L = C^T G^T(GCA + BNC)$. To achieve global stability of this control, define

$$\bar{\gamma} = \gamma - \lambda_{\min}\left(GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^T A^T C^T G^T\right)$$

With $V(x) = x^T Px$, an upper bound on its derivative is given by

$$\dot{V} \leq -\|x\|^2\lambda_{\min}(Q^TQ + 2\varepsilon P) - 2\|s\|^2\bar{\gamma} + 2\|s\|\kappa - 2\alpha$$

Noting that the last three terms form a parabola, their sum is negative if $\alpha > \kappa^2/4\bar{\gamma}$. Thus, if γ and/or α are chosen large enough, then the system is globally asymptotically stable. Again, with the smoothing term, $u_d = s/(\|s\|^2 + \delta)$, the response results in a boundary layer about the switching surface and the state is ultimately bounded with a bound that depends on δ.

Having shown that robustness increases with increasing gains, it should be mentioned that the conditions given for robust stability may be very conservative, and so, robust stability may be attained without increasing the gains over that required in Secs. II.A and II.B. Good response still will be achieved even if the gains chosen are less than that required for robust stability. In particular, it can be shown that the state is ultimately bounded with a bound that decreases with increasing gains.

III. Algorithm and Example

The theorems and corollaries given in the last section suggest two algorithms by which the parameters in the control Eq. (2) can be designed. Both algorithms result in controls so that the feedback system is globally asymptotically stable to the origin and the sliding surface exists on the domain $\{x: Gy = 0 \text{ and } \|x\| \leq \Omega\}$. Algorithm 1 ensures that the reaching condition $s^T\dot{s} < 0$ is satisfied in a region $R_1 = \{x: \|x_k\| \leq \Omega\}$, and the second algorithm ensures that the reaching condition is satisfied in a region $R_2 = \{x: \|x\| \leq \Omega\}$, where $R_2 \subset R_1$. Note that Algorithm 2 may yield smaller gains.

Algorithm 1.

0) Input A, B, C, G_1, and Ω, where G_1 defines the switching surface.

1) Transform G_1 by $G = (G_1CB)^{-1}G_1$.

2) Check if (A, B, GC) is SPR.

3) If (A, B, GC) is SPR, then let $\gamma_1 = 0$. If (A, B, GC) is not SPR, then choose γ_1 such that $(A - \gamma_1 BGC, B, GC)$ is SPR.

4) Decompose GC as $GC = U\Sigma V_1^T$.

5) Let $\gamma_2 = -(0.5)\lambda_{\min}(GCAV_1\Sigma^{-1}U^T + U\Sigma^{-1}V_1^TA^TC^TG^T)$.

6) Choose $\gamma \geq \gamma_{\min}$, where $\gamma_{\min} = \max(\varepsilon, \gamma_1 + \gamma_2)$ and $\varepsilon > 0$ is small.

7) Let $N = -\gamma G - GCAV_1\Sigma^{-1}U^TG$.

8) For $u_d = s/(\|s\| + \delta)$, choose $\alpha \geq \alpha_{\min}$, where $\alpha_{\min} = \|GCA\|\Omega$.

For $u_d = s/(\|s\|^2 + \delta)$, choose $\alpha \geq \alpha_{\min}$, where $\alpha_{\min} = \min\{\|GCA\|^2\Omega^2/4\gamma, \lambda_{\max}(L + L^T)\Omega^2/2\}$ and $L = C^TG^T(GCA + NC)$.

Step 2 can be accomplished by checking if a solution exists for P and Q, using the algorithm in Ref. 14, or by determining whether condition 1 of Lemma 1 is satisfied. Step 3 can be accomplished by increasing γ_1 until the system is SPR. The calculation in step 6 is made to ensure that γ satisfies the requirements in Theorem 1 and in Theorem 2. If robust stability is desired, then steps 6 and 8 can be modified according to the results in Sec.II.C. Detailed design considerations for selecting the scalar parameters Ω, γ, α, and δ are given in Sec. III.A.

In Algorithm 2, steps 0–3 are the same as Algorithm 1. The control is then chosen as $u = Ny - \alpha u_d$, where $N = -\gamma G$, $\gamma \geq \gamma_1$ is chosen such that $\gamma > 0$, and $\alpha \geq \alpha_{\min}$ is chosen as described in step 8 of Algorithm 1.

A. Example

The following example demonstrates Algorithm 1. Consider a single input aircraft example,[1]

$$A = \begin{bmatrix} -0.277 & 1 & -0.0002 \\ -17.1 & -0.178 & -12.2 \\ 0 & 0 & -6.67 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 6.67 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad G_1 = [-0.4635 \quad 1]$$

where the states are angle of attack (deg), pitch rate (deg/s), and elevator angle (deg); the control is the command to the elevator (deg). This system is not SPR, but a linear feedback with $\gamma_1 = 0.7$ results in an SPR system. If Ω is chosen to be 10, then the algorithm yields $\gamma_{\min} = 1.48$. Selecting $\gamma = \gamma_{\min}$, then $N = [0.0428 \quad -0.0924]$ and $\alpha_{\min} = 12$ for $u_d = s/(\|s\| + \delta)$, whereas $\alpha_{\min} = 8$ for $u_d = s/(\|s\|^2 + \delta)$. With γ increased to $\gamma = 50$, $N = [3.41 \quad -7.37]$ and $\alpha_{\min}$ remains at 12 for $u_d = s/(\|s\| + \delta)$, whereas $\alpha_{\min}$ is reduced to 0.704 for $u_d = s/(\|s\|^2 + \delta)$. For comparison, two design methods of El-Khazali and DeCarlo[9] were used on this example to find the linear feedback part of the VSC controller, i.e., the gain N, where the control is rewritten in the form of Eq. (2). When applied to this problem, both methods yielded a gain of $N = [6.3585 \quad -13.7184]$ [defined from Eqs. (38) and (42) of Ref. 9], which is much larger than obtained in this paper.

Algorithm 2 was applied successfully to a third-order example given by Edwards and Spurgeon,[10] with the linear-gain part of the control computed to be $N = [0.6667 \quad 0.3333]$. This is smaller in magnitude than the control computed in Ref. 10, where $N = [1.2442 \quad 0.6221]$.

B. Practical Design Considerations

The design procedure outlined above has four design parameters, which are all scalars: Ω, γ, α, and δ. There are tradeoffs among these parameters that must be considered when designing this control system. The goal of the design is to achieve good robustness and disturbance rejection. Because this is achieved during the sliding phase, the sliding surface should be as large as possible; the reaching time should be small; and the boundary layer should be small. Making Ω, γ, and α large while making δ small achieves this goal. On the other hand, the control signal should be smooth enough to avoid chattering and it should not have large peak values; therefore, γ and α should be reasonably small. Moreover, many actuators do not tolerate high switching levels, even if the switching is somewhat smoothed. Therefore, the size of the discontinuous part of the control, αu_d, should be within an acceptable range. For a control where $u_d = s/(\|s\| + \delta)$, the peak size of αu_d is α in each control channel. However, reducing α for this control requires reducing the size of the sliding surface Ω, thereby reducing the robustness and disturbance rejection properties.

The control $u_d = s/(\|s\|^2 + \delta)$ has an advantage in that the size of the sliding surface can remain large while the size of αu_d is reduced to acceptable levels. For this control, the magnitude of αu_d depends not only on α but also on δ, as seen in the plot of a scalar control u_d vs s in Fig. 1 for $\delta = 0.05$ and 0.005. The peak value of u_d is $1/(2\sqrt{\delta})$ both for a scalar control and for each channel of a multi-input control. The smaller α and the larger δ, then the smaller the peak values of the discontinuous control. It was shown in the preceding section that there is a tradeoff between the sizes of α and γ while maintaining the same size of the control surface Ω. Thus, the discontinuous part of the control αu_d can be reduced in size, and the linear part Ny increased in size without a reduction in Ω. Moreover, the peak value of u_d can be further reduced and good disturbance rejection retained by scheduling δ between a large value far from the surface and a small value close to the surface. Shown in Fig. 2 is the scalar control $u_d = s/(\|s\|^2 + \delta)$ vs s for the case where δ is scheduled between $\delta_1 = 0.05$ and $\delta_2 = 0.001$, i.e., $\delta = \delta_1$ when $\|s\| > 0.75\sqrt{\delta_1}$, $\delta = \delta_2$ when $\|s\| < 0.1\sqrt{\delta_2}$, and δ is linearly interpolated for $0.1\sqrt{\delta_2} \leq \|s\| \leq 0.75\sqrt{\delta_1}$. This scheduling is equivalent to starting with a wide boundary layer and tightening it as the trajectory gets closer to the switching surface.

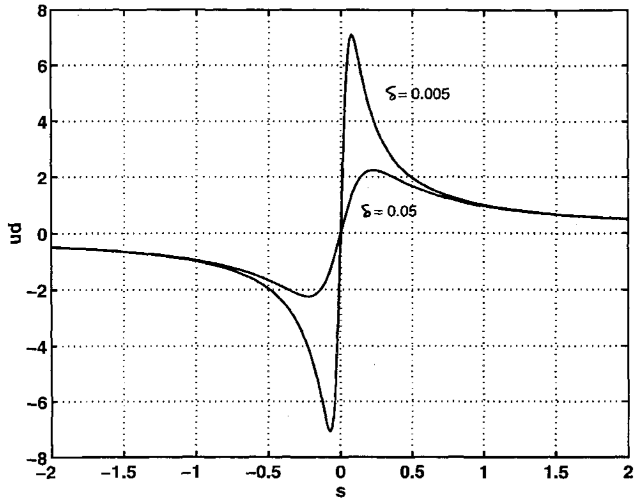

Fig. 1 Scalar control $u_d = s/(\|s\|^2 + \delta)$ for $\delta = 0.05$ and 0.005.

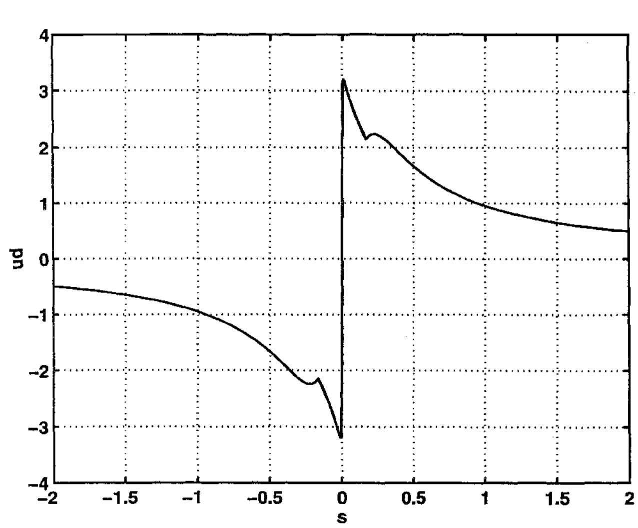

Fig. 2 Scalar control $u_d = s/(\|s\|^2 + \delta)$ with δ scheduled such that $\delta = 0.001$ for $|s| \leq 0.1\sqrt{(0.001)}$, $\delta = 0.05$ for $|s| \geq 0.75\sqrt{(0.05)}$, and δ is linearly interpolated between 0.001 and 0.05 for $0.1\sqrt{(0.001)}|s| < 0.75\sqrt{(0.05)}$.

Thus, good disturbance rejection and good robustness are achieved while avoiding the high peak in u_d of $1/(2\sqrt{\delta}) = 16$ that would result if a constant value of $\delta = 0.001$ is used. This ad hoc scheme for the selection of δ does not affect the reaching condition or the global asymptotic stability proven in Sec. II. Note that this ad hoc scheme also works well for multi-input control.

The above considerations were incorporated into the design of the numerical example described. The control $u_d = s/(\|s\|^2 + \delta)$ is selected, where δ is scheduled between 0.05 and 0.001 as described in the previous paragraph. A selection of $\Omega = 10, \gamma = 50$, and $\alpha = \alpha_{\min} = 0.704$ gave good performance. The response of the system to an initial condition of $[0\ \ 2\ \ 0]^T$ and to a matched disturbance of $h = B[\sin(12t)]$ is given in Fig. 3. The system started sliding at approximately $t = 0.05$ s. Note that the peak control level is acceptable and the disturbance rejection is very good. This choice of u_d works better than $u_d = s/(\|s\| + \delta)$, which results in essentially the same responses for the states but requires an excessive level of discontinuous control, 12 deg. For comparison, consider a linear static control of $u = Ky$, where $K = [3.24\ \ -6.8]$ is chosen to give closed-loop eigenvalues of $-3.06 \pm 3.39j, -46$. The eigenvalues of the sliding-mode equation are $-3.06 \pm 3.06j$, so that the nominal behavior of the linear control and the sliding-mode control are comparable. The response of the linear control system to the initial condition $[0\ \ 2\ \ 0]^T$ with the matched disturbance $h = B[\sin(12t)]$ is given in Fig. 4. Note that the control signal is very similar to the sliding-mode control shown in Fig. 3, but the disturbance rejection is markedly worse than that achieved by the sliding-mode control.

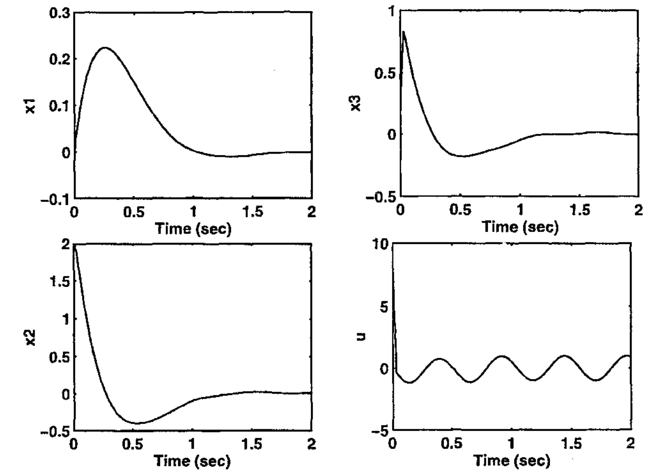

Fig. 3 Response to initial condition of $[0\ \ 2\ \ 0]^T$ and disturbance of $h = B[\sin(12t)]$ for sliding-mode control.

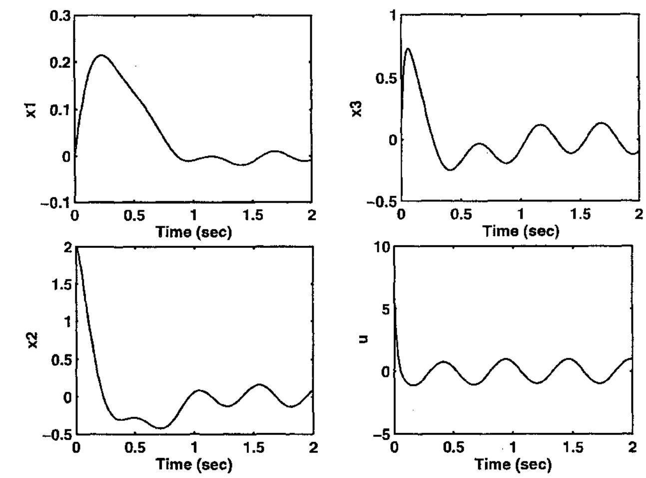

Fig. 4 Response to initial condition of $[0\ \ 2\ \ 0]^T$ and disturbance of $h = B[\sin(12t)]$ for the linear control.

IV. Conclusions

This paper contains a design method to obtain a control that satisfies the reaching condition for VSC with output feedback. By combining a linear feedback with a discontinuous feedback, good disturbance rejection and robustness are achieved without requiring excessive control effort. Because the system is SPR or can be made SPR, the sliding surface is globally reachable with the developed control. There is only one mild restriction for application of this control law beyond that required for the standard output feedback VSC, namely, that the triple (A, B, GC) must be observable. The analysis is developed for two forms of the discontinuous feedback, which allows some freedom in the design. The robustness of the control improves as the gains are increased. A discussion is given in Sec. III.A on how the scalar design parameters Ω, γ, α, and δ can be chosen to achieve good disturbance rejection and robustness with acceptable peak values of the control signal. The results were demonstrated on a third-order system for simplicity; however, the results hold for larger systems with multiple inputs.

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